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ON A DIVISOR PROBLEM RELATED TO THE EPSTEIN ZETA-FUNCTION, II

GUANGSHI LÜ, JIE WU & WENGUANG ZHAI

ABSTRACT. Recently by the theory of modular forms and the Riemann zeta-function, Lü improved the estimates for the error term in a divisor problem related to the Epstein zeta-function established by Sankaranarayanan. In this short note, we are able to further sharpen some results of Sankaranarayanan and of Lü, and to establish corresponding Ω -estimates.

1. INTRODUCTION

For a positive definite quadratic form $Q(\mathbf{y}) = Q(y_1, \dots, y_\ell)$ in $\ell \geq 2$ variables with integral coefficients, we can write it in Siegel's notation as

$$Q(\mathbf{y}) = \frac{1}{2} \mathbf{A}[\mathbf{y}] = \frac{1}{2} \mathbf{y}^t \mathbf{A} \mathbf{y} = \sum_{i < j} a_{ij} y_i y_j + \frac{1}{2} \sum_i a_{ii} y_i^2,$$

where \mathbf{y}^t is the transpose of \mathbf{y} , and the matrix $\mathbf{A} = (a_{ij})$ has integral entries which are even on the diagonal, i.e., $a_{ii} \equiv 0 \pmod{2}$ for $0 \leq i \leq \ell$. Then the corresponding Epstein zeta-function is initially defined by the Dirichlet series

$$(1.1) \quad Z_Q(s) := \sum_{\substack{y_1 \in \mathbb{Z} \\ (y_1, \dots, y_\ell) \neq (0, \dots, 0)}} \cdots \sum_{y_\ell \in \mathbb{Z}} Q(y_1, \dots, y_\ell)^{-s}$$

for $\Re s > \ell/2$. We can also rewrite it as, in the same region,

$$Z_Q(s) = \sum_{n \geq 1} a_n n^{-s},$$

where a_n is the number of the solutions of the equation $Q(\mathbf{y}) = n$ with $\mathbf{y} \in \mathbb{Z}^\ell$. It is known that $Z_Q(s)$ has an analytic continuation to the whole complex plane \mathbb{C} with only a simple pole at $s = \ell/2$, and satisfies a functional equation of Riemann type

$$(d^{1/\ell}/2\pi)^s \Gamma(s) Z_Q(s) = (d^{1-1/\ell}/2\pi)^{\ell/2-s} \Gamma(\ell/2-s) Z_{\overline{Q}}(\ell/2-s) \quad (s \in \mathbb{C}),$$

where d is the discriminant of Q and $\overline{Q}(\mathbf{y}) := \frac{1}{2} \mathbf{y}^t (d\mathbf{A}^{-1}) \mathbf{y}$ (cf. [9]).

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If we write for any integer $k \geq 1$,

$$Z_Q(s)^k = \sum_{n \geq 1} a_k(n) n^{-s},$$

then

$$a_k(n) = \sum_{n_1 \cdots n_k = n} a_{n_1} \cdots a_{n_k}.$$

In particular $a_1(n) = a_n$. It seems interesting to study the asymptotic behavior of the sum $\sum_{n \leq x} a_k(n)$. It is easy to show that its main term is

$$\operatorname{Res}_{s=\ell/2} (Z_Q(s)^k x^s s^{-1}) = x^{\ell/2} P_k(\log x),$$

where $P_k(t)$ is a polynomial in t of degree $k-1$. Then the real hard work is to study the error term

$$(1.2) \quad \Delta_k^*(Q, x) := \sum_{n \leq x} a_k(n) - x^{\ell/2} P_k(\log x).$$

In 1912, Landau [7] proved that for $\ell = 2$, $\Delta_1^*(Q, x) \ll x^{1/3+\varepsilon}$, where and throughout this paper ε denotes an arbitrarily small positive constant. Landau's method can also be applied to treat the general case. In fact his method implies that for $k \geq 1$ and $\ell \geq 2$,

$$\Delta_k^*(Q, x) \ll x^{\ell/2-\ell/(k\ell+1)+\varepsilon}.$$

Later Chandrasekharan and Narasimhan [1] were able to delete the ε in the exponent of x . In [9], Sankaranarayanan improved these classical results by showing that for $k \geq 2$ and $\ell \geq 3$,

$$(1.3) \quad \Delta_k^*(Q, x) \ll x^{\ell/2-1/k+\varepsilon}.$$

Recently inspired by Iwaniec's book [5], Lü [8] was able to improve (1.3) for the quadratic forms of level one (see [5, Chapter 11]). These quadratic forms are defined by $Q(\mathbf{y}) = \frac{1}{2} \mathbf{A}[\mathbf{y}]$ with $\operatorname{diag}(\mathbf{A}) = \operatorname{diag}(\mathbf{A}^{-1}) \equiv 0 \pmod{2}$, where $\operatorname{diag}(\mathbf{A})$ denotes the set of entries on the diagonal of the matrix \mathbf{A} . Moreover we have that $\det(\mathbf{A}) = 1$, \mathbf{A} is equivalent to \mathbf{A}^{-1} , and the number of variables satisfies $\ell \equiv 0 \pmod{8}$. Denote by \mathcal{Q}_ℓ the set of quadratic forms of level one with ℓ variables. For $Q \in \mathcal{Q}_\ell$, we have (see [5, (11.32)] or [8, Lemma 2.1])

$$a_n = A_\ell \sigma_{\ell/2-1}(n) + a_f(n, Q) \quad (n \geq 1),$$

where

$$A_\ell := \frac{(2\pi)^{\ell/2}}{\zeta(\ell/2)\Gamma(\ell/2)}, \quad \sigma_k(n) = \sum_{d|n} d^k,$$

$\zeta(s)$ is the Riemann zeta-function, $\Gamma(s)$ is the Gamma function and $a_f(n, Q)$ is the n th Fourier coefficient of a cusp form $f(z, Q)$ of weight $\ell/2$ with respect to the full modular group $\operatorname{SL}(2, \mathbb{Z})$. Thus

$$(1.4) \quad Z_Q(s) = A_\ell \zeta(s - \ell/2 + 1) \zeta(s) + L(s, f) \quad (\Re s > \ell/2),$$

where $L(s, f)$ is the Hecke L -function associated with $f(z, Q)$. According to Deligne's well known work [2], we have

$$(1.5) \quad |a_f(n, Q)| \leq n^{(\ell/2-1)/2} \tau(n),$$

where $\tau(n)$ is the divisor function. With the help of these properties, Lü proved, by complex integration method, a better estimate than Sankaranarayanan's (1.3) for all $k \geq 3$ and $8 \mid \ell$. For $r \geq 0$, the r -dimensional divisor function $\tau_r(n)$ is defined by

$$\zeta(s)^r = \sum_{n \geq 1} \tau_r(n) n^{-s} \quad (\Re s > 1).$$

The r -dimensional divisor problem is concerned with the estimate of the error term

$$(1.6) \quad \Delta_r(x) := \sum_{n \leq x} \tau_r(n) - \operatorname{Res}_{s=1} (\zeta(s)^r x^s s^{-1}) = \sum_{n \leq x} \tau_r(n) - x G_r(\log x),$$

where $G_r(t)$ is a polynomial of degree $r-1$ if $r \geq 1$ and $G_0(t) \equiv 0$. It is known that

$$(1.7) \quad \Delta_r(x) \ll x^{\theta_r + \varepsilon} \quad (x \geq 2)$$

where

$$(1.8) \quad \theta_0 = 0, \quad \theta_1 = 0, \quad \theta_2 = 131/416, \quad \theta_3 = 43/96$$

and

$$(1.9) \quad \theta_r = \begin{cases} (3r-4)/(4r) & \text{if } 4 \leq r \leq 8, \\ 35/54 & \text{if } r = 9, \\ 41/60 & \text{if } r = 10, \\ 7/10 & \text{if } r = 11, \\ (r-2)/(r+2) & \text{if } 12 \leq r \leq 25, \\ (r-1)/(r+4) & \text{if } 26 \leq r \leq 50, \\ (31r-98)/(32r) & \text{if } 51 \leq r \leq 57, \\ (7r-34)/(7r) & \text{if } r \geq 58. \end{cases}$$

(The case of $r = 0, 1$ is trivial. See [3] for $r = 2$, [6] for $r = 3$ and [4, Theorem 12.3] for $r \geq 4$.) Lü's result (see [8, Theorem 1.2]) can be stated as follows

$$(1.10) \quad \Delta_k^*(Q, x) \ll \begin{cases} x^{\ell/2-1/2+\varepsilon} & \text{if } k = 3, \\ x^{\ell/2-1+\theta_k+\varepsilon} & \text{if } k \geq 4. \end{cases}$$

In this short note, we can further improve Sankaranarayanan's (1.3) with $k = 2$ and Lü's (1.10) with $k = 3$.

Theorem 1. *Let $k \geq 2$ and $8 \mid \ell$. Then for any quadratic form $Q(\mathbf{y}) \in \mathcal{Q}_\ell$, we have*

$$\Delta_k^*(Q, x) \ll x^{\ell/2-1+\theta_k+\varepsilon},$$

where θ_k is the exponent in (1.7).

For comparison, we note

$$\ell/2 - 1 + \theta_k = \begin{cases} \ell/2 - 1/2 - 0.185\dots & \text{if } k = 2, \\ \ell/2 - 1/2 - 0.052\dots & \text{if } k = 3, \end{cases}$$

which are better than (1.3) with $k = 2$ and (1.10), respectively.

For $k = 2$ or 3 , we also can establish Ω -type result.

Theorem 2. *Let $2 \leq k \leq 8$ and $8 \mid \ell$. If there is a positive constant δ such that*

$$(1.11) \quad \theta_r \leq (k-1)/(2k) - \delta \quad (0 \leq r \leq k-1),$$

then for any quadratic form $Q(\mathbf{y}) \in \mathcal{Q}_\ell$ and $\varepsilon > 0$, we have

$$(1.12) \quad \Delta_k^*(Q, x) = \Omega \left(x^{\ell/2-1+(k-1)/(2k)} (\log x)^{(k-1)/(2k)} (\log_2 x)^{\beta_k} (\log_3 x)^{-\gamma_k-\varepsilon} \right)$$

where $\beta_k := (k^{(2k)/(k+1)} - 1)(k+1)/(2k)$ and $\gamma_k := (3k-1)/(4k)$.

In particular (1.12) holds unconditionally for $k = 2$ or 3 .

Our method is different from [8]. First we shall establish relations between $\Delta_k(x)$ and $\Delta_k^*(Q, x)$ and then deduce Theorems 1 and 2 from known O -type and Ω -type estimates for $\Delta_k(x)$.

2. PRELIMINARY LEMMAS

This section is devoted to establish three preliminary lemmas, which will be needed in the proof of Theorems 1 and 2.

Lemma 2.1. *For any $\varepsilon > 0$, we have*

$$\int_1^x \Delta_r(t) dt \ll_{r,\varepsilon} x^{1+\delta_r+\varepsilon} \quad (x \geq 1),$$

where

$$(2.1) \quad \delta_r := \begin{cases} 1/2 - 1/r & \text{if } r = 2, 4, 6, 8, \\ 1/2 - 1/(r+1) & \text{if } r = 1, 3, 5, 7. \end{cases}$$

Proof. By Perron's formula [11, Theorem II.2.3], we obtain, with $b := 1 + 1/\log x$,

$$(2.2) \quad \int_0^x \Delta_r(u) du = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} F_r(s; x) ds - \int_0^x u G_r(\log u) du.$$

where $b := 1 + 1/\log x$ and $F_r(s; x) := \zeta(s)^r x^{s+1} / \{s(s+1)\}$.

Let $\max\{1 - 6/r, 0\} < a < 1$. By using the classical estimate

$$\zeta(s) \ll (|t| + 2)^{\max\{(1-\sigma)/3, 0\}} \log(|t| + 2),$$

we deduce that for all $\varepsilon > 0$ and $T > 0$,

$$\int_{a \leq \sigma \leq b, |\tau|=T} |F_r(s; x)| |ds| \ll (x^2 T^{-2} + x^{1+a} T^{-2+\max\{(1-a)r/3, 0\}}) (\log T)^r,$$

and

$$\int_{\sigma=b, |t| \geq T} |F_r(s; x)| |ds| \ll x^2 T^{-1} (\log T)^r.$$

Using the preceding estimates and shifting the line of integration from $\sigma = b$ to $\sigma = a$, the residue theorem implies that

$$\begin{aligned} \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} F_r(s; x) ds &= \frac{1}{2\pi i} \int_{b-iT}^{b+iT} F_r(s; x) ds + O\left(\frac{x^2}{T} (\log T)^r\right) \\ &= \int_0^x u G_r(u) du + \frac{1}{2\pi i} \int_{a-iT}^{a+iT} F_r(s; x) ds \\ &\quad + O\left(\frac{x^2}{T} (\log T)^r + \frac{x^{1+a}}{T^{2-\max\{(1-a)r/3, 0\}}} (\log T)^r\right), \end{aligned}$$

where we have used the relation

$$\operatorname{Res}_{s=1} (F_r(s; x)) = \int_0^x u G_r(u) du.$$

Making $T \rightarrow \infty$ and inserting the formula obtained into (2.2), we find that

$$\begin{aligned} \int_1^x \Delta_r(u) du &= \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} F_r(s; x) ds \\ &\ll x^{1+a} \int_{-\infty}^{+\infty} \frac{|\zeta(a+it)|^r}{(|t|+1)^2} dt. \end{aligned}$$

When $r = 2, 4, 6, 8$, the last integral is convergent for any $a > 1/2 - 1/r \geq \max\{1 - 6/r, 0\}$ (see [4, Lemma 13.1 and Theorem 13.4]). For $r = 2k - 1$ ($1 \leq k \leq 4$), we have

$$\int_{-\infty}^{+\infty} \frac{|\zeta(a+it)|^r}{(|t|+1)^2} dt \leq \left\{ \int_{-\infty}^{+\infty} \frac{|\zeta(a+it)|^{2(k-1)}}{(|t|+1)^2} dt \right\}^{1/2} \left\{ \int_{-\infty}^{+\infty} \frac{|\zeta(a+it)|^{2k}}{(|t|+1)^2} dt \right\}^{1/2} < \infty$$

provided $a > 1/2 - 1/(2k) = 1/2 - 1/(r+1) \geq \max\{1 - 6/r, 0\}$. This completes the proof. \square

Lemma 2.2. *For $r \geq 0$, we have*

$$(2.3) \quad \sum_{n \leq x} \tau_r(n) n^{\ell/2-1} = x^{\ell/2} G_r^*(\log x) + x^{\ell/2-1} \Delta_r(x) + O(x^{\ell/2-1+\delta_r}),$$

where $G_r^*(t)$ is a polynomial of degree $r - 1$ with the convention that $G_0^*(t) \equiv 0$ and the constant $\delta_r \geq 0$ is given by (2.1). In particular

$$(2.4) \quad \sum_{n \leq x} \tau_r(n) n^{\ell/2-1} = x^{\ell/2} G_r^*(\log x) + O(x^{\ell/2-1+\theta_r}).$$

Proof. With the help of (1.6) and Lemma 2.1, a simple partial summation yields

$$\begin{aligned} \sum_{n \leq x} \tau_r(n) n^{\ell/2-1} &= \int_1^x t^{\ell/2-1} (t G_r(\log t))' dt + \int_{1-}^x t^{\ell/2-1} d\Delta_r(t) \\ &= x^{\ell/2} G_r^*(\log x) + x^{\ell/2-1} \Delta_r(x) + O(x^{\ell/2-1+\delta_r}). \end{aligned}$$

This completes the proof. \square

In order to state our third lemma, it is necessary to introduce some notation. By (1.4), we can write, for $\Re s > \ell/2$,

$$\begin{aligned} Z_Q(s)^k &= \sum_{0 \leq r \leq k} A_\ell^r C_k^r \zeta(s)^r L(s, f)^{k-r} \zeta(s - \ell/2 + 1)^r, \\ \zeta(s - \ell/2 + 1)^k &= \sum_{0 \leq r \leq k} A_\ell^{-k} C_k^r (-1)^{k-r} \zeta(s)^{-k} L(s, f)^{k-r} Z_Q(s)^r. \end{aligned}$$

These imply that

$$(2.5) \quad a_k(n) = \sum_{0 \leq r \leq k} A_\ell^r C_k^r \sum_{dm=n} b_{k,r}(d) \tau_r(m) m^{\ell/2-1},$$

$$(2.6) \quad \tau_k(n) n^{\ell/2-1} = \sum_{0 \leq r \leq k} (-1)^{k-r} A_\ell^{-k} C_k^r \sum_{dm=n} c_{k,r}(d) a_r(m),$$

where $b_{k,r}$ and $c_{k,r}$ are defined by the relation

$$\zeta(s)^r L(s, f)^{k-r} = \sum_{n \geq 1} b_{k,r}(n) n^{-s}, \quad \zeta(s)^{-k} L(s, f)^{k-r} = \sum_{n \geq 1} c_{k,r}(n) n^{-s},$$

for $\Re s > \ell/2$.

Lemma 2.3. *Let $j \geq 0$, $k \geq 2$, $0 \leq r \leq k$, $8 \mid \ell$ and $\theta > (\ell + 2)/4$. Then for any quadratic form $Q(\mathbf{y}) \in \mathcal{Q}_\ell$ and $d_{k,r} = b_{k,r}$ or $c_{k,r}$, we have*

$$(2.7) \quad \sum_{n \leq x} \frac{|d_{k,r}(n)|}{n^\theta} \ll_{j,\ell,\theta} 1 \quad (x \geq 2),$$

$$(2.8) \quad \sum_{n \leq x} \frac{d_{k,r}(n) (\log n)^j}{n^\theta} = C_f(j, k, r, \theta) + O(x^{-\theta + (\ell+2)/4 + \varepsilon}) \quad (x \geq 2),$$

where $C_f(j, k, r, \theta)$ is a constant.

Proof. By the definition of $b_{k,r}$ and $c_{k,r}$, we have

$$\begin{aligned} b_{k,r}(n) &= \sum_{d_1 \cdots d_r m_1 \cdots m_{k-r} = n} a_f(Q, m_1) \cdots a_f(Q, m_{k-r}), \\ c_{k,r}(n) &= \sum_{d_1 \cdots d_k m_1 \cdots m_{k-r} = n} \mu(d_1) \cdots \mu(d_k) a_f(Q, m_1) \cdots a_f(Q, m_{k-r}). \end{aligned}$$

We treat only the case of $b_{k,r}$ and the latter is completely similar. With the help of the Deligne inequality (1.5), we have

$$\begin{aligned} \sum_{n \leq x} |b_{k,r}(n)| &\leq \sum_{d \leq x} \tau_r(d) \sum_{m \leq x/d} \tau_{2(k-r)}(m) m^{(\ell-2)/4} \\ &\leq \sum_{d \leq x} \tau_r(d) (x/d)^{\ell/4+1/2} (\log x)^{2k-2r-1} \\ &\ll_{j,\ell} x^{(\ell+2)/4} (\log x)^{2k-2r-1} \quad (0 \leq r \leq k). \end{aligned}$$

From this, a simple partial integration allows us to deduce (2.7) and (2.8). \square

3. PROOF OF THEOREM 1

By (2.5), (2.4) of Lemma 2.2 and (2.7) of Lemma 2.3, it follows that

$$\sum_{n \leq x} a_k(n) = x^{\ell/2} \sum_{0 \leq r \leq k} C_k^r A_\ell^r \sum_{d \leq x} \frac{b_{k,r}(d)}{d^{\ell/2}} G_r^*(\log(x/d)) + O(x^{\ell/2-1+\theta_k+\varepsilon}).$$

Since $\ell/2 > (\ell+2)/4$, (2.8) of Lemma 2.3 implies that

$$\sum_{0 \leq r \leq k} C_k^r A_\ell^r \sum_{d \leq x} \frac{b_{k,r}(d)}{d^{\ell/2}} G_r^*(\log(x/d)) = P_k(\log x) + O(x^{1/2-\ell/4+\varepsilon}).$$

Inserting it into the preceding formula, we get the required result. \square

4. PROOF OF THEOREM 2.

From (2.6) and (1.2), we can deduce that

$$\begin{aligned} \sum_{n \leq x} \tau_k(n) n^{\ell/2-1} &= x^{\ell/2} G_k^*(\log x) + O(x^{(\ell+2)/4+\varepsilon}) \\ &\quad + \sum_{0 \leq r \leq k} (-1)^{k-j} A_\ell^{-k} C_k^r \sum_{d \leq x} c_{k,r}(d) \Delta_r^*(Q, x/d), \end{aligned}$$

where we have used the following estimate

$$\sum_{0 \leq r \leq k} (-1)^{k-j} A_\ell^{-k} C_k^r \sum_{d \leq x} c_{k,r}(d) (x/d)^{\ell/2} P_r(\log(x/d)) = x^{\ell/2} G_k^*(\log x) + O(x^{(\ell+2)/4+\varepsilon}).$$

Comparing with (2.3) of Lemma 2.2 yields

$$x^{\ell/2-1} \Delta_k(x) = \sum_{0 \leq r \leq k} (-1)^{k-r} A_\ell^{-k} C_k^r \sum_{d \leq x} c_{k,r}(d) \Delta_r^*(Q, x/d) + O(x^{\ell/2-1+\delta_k}).$$

Under hypothesis (1.11), by (2.5), (2.3) of Lemma 2.2 and (2.7) of Lemma 2.3 we have

$$\begin{aligned} \Delta_r^*(Q, x) &\ll x^{\ell/2-1+r/\{2(r+1)\}-\delta+\varepsilon} \\ &\ll x^{\ell/2-1+(k-1)/(2k)-\delta/2} \end{aligned}$$

for $0 \leq r \leq k-1$. Inserting into the preceding formula and using (2.7), we can deduce

$$(4.1) \quad x^{\ell/2-1} \Delta_k(x) = A_\ell^{-k} \sum_{d \leq x} c_{k,k}(d) \Delta_k^*(Q, x/d) + O(x^{\ell/2-1+(k-1)/(2k)-\delta/2}).$$

On the other hand, according to Soundararajan [10], we have, for any $k \geq 2$,

$$(4.2) \quad \Delta_k(x) = \Omega((x \log x)^{(k-1)/(2k)} (\log_2 x)^{\beta_k} (\log_3 x)^{-\gamma_k}).$$

Now on noting (2.7) of Lemma 2.3, the first assertion of Theorem 2 follows from (4.1) and (4.2).

Finally in view of (1.8), it is easy check that the hypothesis (1.11) is satisfied when $k = 2$ or 3 . Therefore (1.12) holds unconditionally for these two values of k .

This completes the proof of Theorem 2.

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